VIRTUAL EXPERIMENTING AS A TOOL FOR TROUBLE-FREE ECONOMIC ANALYSIS

(BENEFITS OF CI ASSISTED SIMULATIONS FOR UNDERSTANDING COMPLEXITIES IN ECONOMICS)

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Virtuálne experimentovanie ako nástroj bezproblémovej ekonomickej analýzy
(Prínosy CI podporovaných simulácií na pochopenie zložitých problémov v ekonómii)

Abstract**:** *Computational Intelligence (CI) assisted simulations in socio-economic reasoning bring several useful benefits in understanding complexities for scholars and students, too. Virtual experimenting in digital laboratories allows a wider approach to economic knowledge for students and everybody who should understand complex economic phenomena. For the purpose of illustrating economic contents, the author used an excellent paper by Italian scholars Chiarella, Dieci, Gardini on asset price dynamics. However, the author's main goal is to show how ITC devices and theoretical informatics, computational intelligence, software engineering, and advanced cognitive sciences as new tools and approaches may help understand complex economic phenomena to wider economic community. He is convinced that such devices, methods and tools have promising future in terms of familiarising a wider community with economic knowledge.*

Keywords**:** c*omplex dynamics, genotype-map, global dynamics, heterogeneous agents, phenotype entity, qualitative theory, non-invertible maps, speculation, trouble freeing*

JEL Classification: C 61, D 84, G 12

Expositional introduction

In general economics, but also in other branches of economic science, there frequently arise problems whose analysis, consecutive exposition and finally understanding and mastering need good preliminary mathematical groundwork and solid skills in using suitable tools of mathematical analysis. These needs have been arising in economics for a long time because economists have embedded

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assumptions, sometimes even prejudgments into their mental models, which require good knowledge of mathematics and skills in the application of methods of mathematical analysis and mathematical formalization of such models. This ability was used in direct mode for the first time by A. A. Cournot, W. S. Jevons, and others in the nineteenth century, and it evolved into modern economics. Since economic problems we have in mind here are linked with the study of their nonlinear dynamics, and/or of rather complex evolution, which is, as a matter of fact, a *complex expressis verbis*, indispensible tools of analysis come from demanding mathematical branches. Among others, there are such as nonlinear differential and difference equations, functional analysis and in the practical attaining such knowledge first of all from mathematical theory of dynamic systems and topology. Even at first sight, it is clear that such dispositions are not given to the entire spectrum of individuals, who belong to the community concerned, that is, those individuals who ought to earn knowledge of economic problems in question. These abilities are some kind of monopoly of members of a closed group of economists and appear to be refused to the rest of the given community, although its members might appear to have the knowledge because they "learned it or even have learned it". However "to have learned" does not have to always imply also that the individual has understood "it", understands "it", and it need by no means be a guarantee the individual has mastered the subject-matter so well that they are able to utilise it in a creative way for solving real tasks, that is, they have attained professional skills.

Sometimes, in common life, these dissonances in the deep understanding of complex economic problems and responsibilities for solving new tasks may be an important handicap for economists or political economists as well as for graduate university graduates, too. Particular examples from our and from other post-communist countries certainly need not be described here. Because of the dissonances, there arises a problem of how to help to override them, how to make the knowledge and skill required accessible also for other members of wider economist community, or in other words, how to make it easier for them to approach to these demanding accessible knowledge. Fortunately, nowadays we can certainly answer these questions: the promising methods and tools are the construction of virtual laboratories from up to down and experimentation in them. We owe for these possibilities to advanced ICT, software engineering, and to computational intelligence (CI) on the one hand and on the other one, to the initiative of some theoreticians and methodologists of economic science, who develop modes and methods for using mentioned possibilities, or they directly develop virtual laboratories for significant branches of virtual experimentation.

When Henry Poincaré was working on his *qualitative theory of differential equations,* he barely could imagine how and what widespread methods of analysis of models evolution assisted by ICT (obviously by PC) and by new methods of programming will become. Here we have in mind his finding that it is possible to look at differential equations as some systems that "are walking" in time, that is, time is "ticking away" like as a hand on the clock showing seconds. Nowadays, this

manner of imaging is referred to as "strobing", which is a method of recording the earned value of variable only in a particular instance of time, that is in discrete time. Such representation is called in professional terminology as "the Poincaré section". In the context of this paper, due to Poincaré, the innovation concerned is useful at least for three reasons:
– *First,* PC is computing in discrete time;
– *Second*, under experimental measurement, it is possible to measure only a finite number of values and first of all;
– *Third*, several economic systems suggested in contemporary economics are in their substance *discrete dynamic systems*[2], if we approach them in a slightly simplified way. However, in reality the economies are discrete-event or successive organisms evolving in inner space of a complex nets.

On the earlier explained methodical foundations there are based procedures, which facilitate the cognition of complex economic systems large community of economists, and should assist mainly to university students. In this way, the cognition process is not bound only to a limited part of this community, which is equipped with higher skills in mathematical analysis. We may regard them as a certain form of facilitating the cognition process in the mainstream economics.

An overall goal of this paper is to show new possibilities (naturally, as long as it is possible in "a rigid/ deadened" form caused by printing on paper) and to compare them with conventional mathematical procedures in order to highlight the differences in demands, which are required by the cognition entity. The secondary goals include then the representation of specific attitudes, methods and tools that are usable for experimentation and on the other hand, experimenting with the selected economic models with the attributes of complex nonlinear systems.

In this paper we present in particular:
– *Routines* (trajectories in time, trajectories in two-dimensional domain of variables, cobweb trajectory of single domain (1D) variable, bifurcation map of a single and of double domain (2D) variables with one and two parameters (parametric basins of attraction), attraction basin of two variables, cycles, manifolds, Lyapunov's exponent of single and two variables, absorbing areal for creation critical curves, and pinpoint attractors first of all in environment of iDMC software, and some of partial attachments for presented models in Excel;
– There are several *models with uncertain future*. Here are some examples:

1. Model of capital market with two types of broker on the capital market, that is buyer and seller (fundamentalists and/or chartist);

2. Decision-making model of a monopolist on prices and amounts that he/her wishes to deliver to an unknown market;

3. Model of adaptation of Cournot's players;

4. Macroeconomic dynamics with adaptive learning and some others, which we might want to analyze for the given purposes. To shorten this paper, we deal in

[2] In this paper we view economic systems exclusively only as nonlinear dynamical systems in discrete time.

greater detail with the first meant model. We shall namely show the analysis of ChiDiGar (Chiarella-Dieci-Gardini) model and the Dieci model. These and several other Italian scholars have a great deal of merit on the progress of system dynamics economic models.

1. Mental Model: Capital Market in Alternative Types of Traders' Expectations

One of the best known cases of imperfect information on performance in the market is capital market. The traders on capital market (hereinafter only traders) cannot know new (future) prices in advance, which is, as matter of fact, a base for speculative trades and brokerage on the Stock Exchange. The so-called textbook problem is choosing the treatment or method of estimation and/or expectation of the future evolution of prices in the capital market. Such task is however, very complicated at first sight, so to say professionally *complex* (term *in sensu stricto*) because of the third subject is involved in the performance, that is the market administrator. These complexities are the reason why the fable of capital market or directly mental model one is suitable for arguments supporting the claims in this paper on the possibility of earning economic knowledge easier for a wider collection of persons concerned.

Such fabula may be as follows. Let us we have two different *populations of traders* with capital stock X and Y. The differences in this product of decision making can be written as sub index i, $i \in \{f, c\}$, where f ... are *fundamentalists*, and c ... are *chartists*. Let us begin with the idea and/or consideration that P_t is the logarithm of price of capital stocks in time t. It bears in mind that in each time instance t, and in each population of traders i, their assets are invested for risk stock certificates, they are in this way, entities of the "risk bearing" type (they love risks). Let symbol $\Omega_{i,t}$ refers to assets of trader i, and symbol $Z_{i,t}$ refers to the share of this asset, which the trader i is to decide to invest into risk stock certificate in time t. Now it is coming on order the possibility to clarity of the term syuzhet as the manner of realisation of having in mind that fabula story.[3] It is as follows. The assets of trader having in mind in discrete time may be written in coming step of time ($t+1$) as the equation

$$\Omega_{i,t+1} = \Omega_{i,t} + \Omega_{i,t} (P_{t+1} - P_t), \quad (1)$$

where product of difference $\Omega_{i,t} (P_{t+1} - P_t)$ in (1) is the yield from risk purchase.[4] This fabula-syuzhet character of such models is the reason why they are very suitable for *computational story-telling*, for example in program STELLA. The apparent problem of a market subjects in the relationship (1) is an uncertainty of logarithm of future price P_{t+1}, which they must estimate by certain mode. However, just then this

[3] The fabula and syuzhet are two of the most basic and most important narrative concepts that story builders have at their disposal, yet few in economics know exactly what they mean: The fabula as the global perspective of the story and the syuzhet, on the other hand, represents the subjective, ground-level discombobulating (to throw into a state of mental uncertainty) of the fabula, see [8].

[4] In the other, a not very complicate explanation, we have decided for that stage to leave out the possibility of alternative buying of secure certificates, for example bonds and to focus only on the risk decision of both type of traders. In other words, we have omitted a priori risk aversion from the psychic of subjects.

mode of estimation is in what that populations differed from one another.

Let we assume that every trader from population *i* calculates conditional *average* of $E_{i,t}(\Omega_{i,t-1})$ and *variance* of $V_{i,t}(\Omega_{i,t+1})$ respectively in relation (1) as assuming that the difference $(P_{t+1}-P_t)$ is conditionally normal so

$$E_{i,t}(\Omega_{i,t-1}) = \Omega_{i,t} + \Omega_{i,t} E_{i,t}(P_{t+1}-P_t), \tag{2}$$

$$V_{i,t}(\Omega_{i,t+1}) = \Omega^2_{i,t} V_{i,t}(P_{t+1}-P_t), \tag{3}$$

Without further explanation in detail of existing relations, let us directly assume that fundamentalists calculate on the base of the idea that prices of certificates are copying average reverse process of basic value of average evolving in the long-term horizon. This is the reason why they calculate in the sense that an expected earning apposition is proportionate to the difference between logarithm of current price of certificates and of basic value of them that is written by equation

$$E_{f,t}(P_{t+1}-P_t = \eta(W_t-P_t), \tag{4}$$

where η is the speed of average reverse was estimated by fundamentalists, and W_t is the logarithm of basic value in a time *t*. Traders also assume that conditional variance on price change is constant so it is

$$V_{f,t}(P_{t+1}-P_t) = v_f .$$

On the earlier stated assumptions, a fundamentalist demand, in terms of returns, is thus recorded as follows

$$D_{ft} = a(W_t - P_t), \tag{5}$$

where $a\ (=\eta/v_f > 0)$ is the potential of demand of s fundamentalists. If logarithm of price ratio P_t under (upon) of logarithm expected basic value W_t then fundamentalists are trying to buy (sell) a given ratio, because they imagine that the ratio is undervalued (overvalued), so its price is going to increase (diminish). This approach we also use in the case of chartists.

The chartists are assumed that they are not able to bear the given cost structure needed for achieving the information on basic value, which is however achievable for fundamentalists. For that reason, they found their estimates on the average and variance of expected earnings from risk investments and so on data which do not need costs contained in recent return (log price) changes.[5] If we use $\psi_{t,t+1}$ to denote the chartists' expectation at time *t* of the log price change (i.e. return) over the next trading period, i.e. so in math formalism, we can write that as

$$\psi_{t,t+1} = E_t(P_{t+1}-P_t) = E_t(P_{t+1}) - P_t . \tag{6}$$

In other words, chartists are assumed that they calculate $E_{c,t}(P_{t+1}-P_t)$ based on extrapolated past price changes according to the simple adaptive scheme

$$\psi_{t,t-1} = \psi_{t-1,t} + c(P_t - P_{t-1} - \psi_{t-1,t}) \quad , \tag{7}$$

where *c* (*c* is from interval $0 < c \leq 1$) is the speed at which they adjust their estimate of the trend to the most recent price changes. Alternatively, the quantity

[5] Alternatively, we can imagine the chartists in the way that they trust more in statistical estimation of past trends of prices than to the estimation of basic value. Symbol E was used from verb "Estimation". It is however suitable also for the term "Expectation", unfortunately.

$\tau = 1/c$ may be viewed as the time lag in chartists' information.[6] In such case, the chartist asset demand is given by
$\zeta_{c,t} = \frac{(\psi_{t,t+1}) - g_t}{\alpha_c v_c}$. Unlike the fundamentalists, the chartists change their estimate of v_c the conditional variance according to the magnitude of. $|(\psi_{t,t+1}) - g_t|$ As this quantity becomes larger, they expect greater volatility and increase their estimate v_c, hence lowering the slope of their demand function. This behaviour leads to the flattening-off of the slope of chartist demand as $|(\psi_{t,t+1}) - g_t|$ becomes larger. The chartist asset demand may thus be characterised in general and may be written by earlier meant authors as

$$D_t^c = h(\psi_{t,t+1} - g_t), \tag{8}$$

where the function *h(.)* has these four properties:
(1) $h'(x) > 0 (\forall x)$,
(2) $h(0) = 0$,
(3) There exists such value *x** that $h''(x) < 0 (> 0)$ for all $x > x^* (< x^*)$, and lastly $\lim_{x \to \pm\infty} h'(x) = 0$

Under certain condition, the function *h* (.) in term (8) for computational simulations may be used explicitly as follows: $h(x) = \gamma arctan x$. This function may be considered as a manner by which "learning" process is introduced to behaviour of chartists. However, it is important to remark that the qualitative analysis performed in the following sections (as also the qualitative dynamics) is not affected by a change of function, because these mainly depend on the properties of *h* (.) given above. Thus, total excess demand for the asset at time *t* is possible to simplify by introducing constancy of W_t, thus $W_t = W$, and assuming $g_t = g$ too, thus we obtain

$$D_t = a(W - P_t) + h(\psi_{t,t+1}) \tag{9}$$

It remains to show of adjustment process of price ratio on the market of assets that are influenced by a *market maker* so to say whose function is to set excess demand to zero at the end of each trading day. They could use for example the following assumptions to describe activities of the market maker and traders in every future day of trading (*t+1*) algorithms):

(1) at the beginning of day *t* the market maker announces the (log) price P_t for that day; (2) the market participants then form excess demand D_t according to (9); (3) the market maker, observing the excess demand, takes a long or short position M_t (by adjusting his/her inventory of assets) in order to clear the market, i.e. such that $D_t + M_t = 0$; (4) the market maker then announces, at the beginning of the next trading period, the new (log) price P_{t+1} calculated as the previous (log) price plus some fraction of the excess demand of the previous presumptuous period, according to $P_{t+1} = P_t + b_p D_t (b_p > 0)$. The process then repeats itself.[7]

[6] These restrictions c ≤ 1, t. j. τ ≥ 1, have simple economic interpretation lying in the fact that chartists cannot review their value estimates ψt,t+1 more frequently than by the state how frequently they receive information on price changes. In such case the frequency is equal to one unit of time.

[7] It is correct to remark that despite of significant complexity of mentioned theoretical presumptuous this theoretical background is excessive simplification in comparison to other theories, which we can find in

Thus, at the beginning of day *(t+1)* the following dynamic adjustments occur following dynamical adjustments made by mutual reciprocal reactions of the market maker and chartists, respectively (it is going actually on steps in the sense of iteration process, that is on difference game):

$$\begin{cases} P_{t+1} = P_t + \beta_p \left[a(W - P_t) + h(\psi_{t,t+1} - g) \right], \\ \psi_{t+1,t+2} = (1 - c)\psi_{t,t+1} + c\beta_p \left[a(W - P_t) + h(\psi_{t,t+1} - g) \right]. \end{cases} \tag{10}$$

According to (10), the evolution of price in time and expectation of chartists can be acquired by by iteration of 2D nonlinear map *Q*, that is $Q: (\psi, P) \rightarrow (\psi', P')$

$$Q: \begin{cases} \psi' = (1-c)\psi + c\beta_p\left[a(W - P_t) + h(\psi)\right] \\ P' = P + \beta_p\left[a(W - P_t) + h(\psi)\right] \end{cases}, \tag{11}$$

where the apostrophe denotes the unit time advancement operator. In accord with (11) in equilibrium the fundamentalists are clear buyers (i.e. investors, because price is under their expected average in long period); on the other hand, the chartists in same situation are clear sellers.[8] By introducing the log price deviation $p = P - \overline{P}$ by substitution, we obtain the new map $T: (y, p) \rightarrow (y', p')$ given in the form

$$T: \begin{cases} \psi' = (1-c)\psi - c\beta_p\left[ap - k(\psi)\right], \\ p' = p - \beta_p\left[ap - k(\psi)\right] \end{cases} \tag{12}$$

where, $\begin{cases} P_{t+1} = P_t + \beta_p \left[a(W - P_t) + h(\psi_{t,t+1} - g) \right], \\ \psi_{t+1,t+2} = (1 - c)\psi_{t,t+1} + c\beta_p \left[a(W - P_t) + h(\psi_{t,t-1} - g) \right] \end{cases}$ and having the origin *O* = *(0, 0)* as unique fixed point.

When we take into account the former introduced *function of self-learning,* that is of formula, $h(.) = \gamma arctan(.)$, we can rewrite (12) to

$$T': \begin{cases} \psi' = (1-c)\psi - c\beta_p\left[ap - (\gamma \arctan(\psi))\right] \\ p' = p - \beta_p\left[ap - (\gamma \arctan(\psi))\right] \end{cases} \tag{12′}$$

and in the following process can use this formula for the creation of virtual laboratory in program iDMC. All that is however a great simplification made in other to reach better understanding, which is another way of a highly demanding explanation.

A well-known Australian economist with Italian roots Carl Chiarella together with his Italian colleagues[9] in their essay on *speculative behaviour of agents* in a *complex capital market* [2] globally analyses in the model dynamics that they created for that purpose. Their model is more professional, however, at the expense of introducing further demands on mathematical skill of the cognition entity. Likewise an earlier

contemporary economic literature in the theoretical area concerned.

[8] It need not be of course taken into account too seriously because of strong simplifications introduced to the story as a whole.

[9] We have in mind Professor Laura Gardini (University of Parma) and Professor Roberto Dieci (University of Urbino). They are excellent proponents of new branch of economic-mathematical methods using computational simulations. Professor Carl Chiarella worked at the Technical University of Sydney and for long time cooperated with those scholars.

described model represents the *capital market* with two populations of traders with capital certificates, namely:

– *fundamentalists,* who base their trade decisions on expected earnings from fundamental (basic) values of portfolio, and

– *chartists*, who base their trade decisions on analysing former price trends.[10]

It is needed to emphasize immediately that the task proposed is of a strong *complex character* and places highly challenging skills in the branch of mathematical analysis. The authors of the paper pay attention to deeper mathematical analysis. However, for reasons of earlier defined aims of our investigation, we are not going to pursue them, and instead focus our attention on the construction of *virtual laboratory* inspired by them and on *experimenting* in the laboratory. It is impossible to forget that also these authors had to make concessions because of complexity of the problematic. For that purpose, they reduced their model to *2D map* which *global dynamical behaviour* on which they conducted a detail mathematical analysis. Just this task we use to show that via computational experimentation in further virtual laboratory, which we constructed in software iDMC for that demanding task according to their theory, may be extended to reach those who gain a better approach to understanding the complex problematic even without having to master highly demanding procedures of mathematical analysis. Two *complex activators* affect the dynamics acquired[11] They are: on the one hand, it is a *parameter, which measures the extent of demand* of the fundamentalists and – on the other hand, it is a *parameter of velocity of chartists' adjustment of their price expectations* of future trend against past changes of prices. *Parametric space* is specific: it is characterized on the one hand by the viewpoint of *local stability* and/or *instability of point of equilibrium* (fix point) and on the other hand, from the point of view of *the map non-invertibility*. For the solution of that problem it is appropriate to use the experimentation method in *absorbing areal,* in which it is possible to generate separate *critical curves* by iteration and also to create *attractor* (or attractors), that is to project one over the other. *The approach and methods of critical curves of noninvertible maps*, as we showed earlier is good assistant in overwriting and in understanding of *global bifurcation* range, which can emerge in evolutionary process. Global bifurcations of higher order can in certain circumstance imitate as if it was stochastic process; however, they still remain in the determinist area. This can also be regarded as contribution to better understanding of the complexity of economic systems and differentiate between deterministic and stochastic aspects in evolution of them because as a rule, if the statistical sequence of some economic variable is rugged, it is prevailingly regarded (by means of thought cut), for stochastic process, although it need not be always. Several, from the outside rugged processes are in fact deterministic, which is demonstrated by our experiments.

[10] Demand of chartists is formed into S-shape by introducing the function arcus tangent (atan). Its aim will be explained later.

[11] Perturbations which are introduced into evolution of system by these acts may be understood as some technique of control; consequently, these parameters are the control parameters of the process.

2. Exhibition of some Mathematical Methods of Genotypic Model Behaviour Study

It is coming to be natural that in order to better understand problems, it is better to begin with the presentation of our laboratory primarily based on graphical outputs from realised experiments. After that, we have to introduce laboratory as a model created in iDMC software, created in the LUA language, which is running Java program. But the model concerned is built on earlier presumptions on capital market, while a current reader need not be familiar with them.[12] After all, theoretical initiative is going back as late as to the original model created by E. C. Zeeman [9], who as early as in the mid seventies was the first one to use for the description of and solution of this task the *mathematical theory of catastrophe.*[13] Although this routine is not very correct; certainly, it would be desirable to present a holistic theoretical construction on which this task is based. Accordingly, it would have to start from least in short with the theory and/or its form presented by the authors mentioned before [2], which is however not possible also due to its very large breadth and complexity of mathematical formalisms. For the purpose of deeper interesting, we recommend readers to study suitable literature on the methods and tools, for example such as in [3]. However, in order to gain an idea of mathematical demands of the task, also in this place, we present you some important passages from the ChiDiGar model.

2.1. Passage: Inquiry on local stability conditions

Let us remind ourselves map *T* written by formulae (12), to gain an easier insight into the problem, or its modification of map *T´* written by formulae (12´). Local stability of fixed point *O* depends from Jacobian of map[14] *T*,

$$DT(\psi, p) = \begin{bmatrix} 1-c+c\beta_p \dfrac{\gamma}{1+(\psi-g)^2}, & -ac\beta_p \\ \beta_p \dfrac{\gamma}{1+(\psi-g)^2}, & 1-a\beta_p \end{bmatrix},$$

[12] We have to repeat that the focus of the present paper is to demonstrate that even highly complex economic problems can be explained in an easier and efficient manner. The problem concerned serves only as a medium for realisation of this approach. The difficulties rest in the fact that without the former knowledge about basic problematic it is impossible to understand the things mentioned nor via using perfect experiments in virtual laboratory.

[13] The author of this paper published several ideas on the possibilities of these treatments in the analysis of economic systems in the Ekonomický časopis/Economic Journal of the Slovak Academy of Sciences. Even with the delay of many years, it can be said that these ideas have not fallen into a fertile soil in Slovak and Czech communities of economics scholars and students. Consequently, only great enthusiasts may hope in this context that common economists possess needed knowledge and practical skill.

[14] Note that in Jacobian for the assumed functional form of h as a function is derivative k´(ψ)=γ/[1+(ψ-g)2].

in *(0,0)*. Let us write for fixed point O by symbol *Tr (Trace,* <in Slovak *stopa*>*)*[15] and by symbol *Det determinant DT (0, 0)* and by formula $\mathcal{P}(z) = z^2 - Tr\, z + Det$, *associated characteristic polynomial* in (13). In this branch of mathematics, the necessary and sufficient condition to have all the eigenvalues of $\mathcal{P}(z)$ less than *1* in absolute value (and thus a locally attracting fixed point) consists of the inequalities:[16]

$$\mathcal{P}(1)=1-Tr+Det>0,\ \mathcal{P}(-1)=1+Tr+Det>0,\ \mathcal{P}(0)=Det<1. \qquad (13)$$

For that map it is possible the conditions (13) rewrite as 2D inequalities:

$$\begin{cases} a\beta_p(2-c) < 2(2-c) + 2c\beta_p k'(0), \\ a\beta_p(1-c) > c\left[\beta_p k'(0) - 1\right] \end{cases} \qquad (14)$$

Figure 1

The graph of two curves

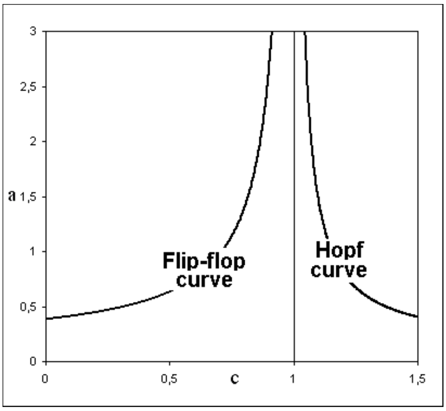


Fig. 1, which we created in Excel, exhibits an area of local stability of origin in parameter space *(c, a)*, while they are from interval $0<c\leq 1,\ a>0$. It follows from inequalities (14) that if starting from parameters *(c, a)* in inner area of stability, the loss of stability emerges either as bifurcation of double period (*flip bifurcation*), and it occurs when the curve is overpassed

$$a = \frac{2}{\beta_p} + \frac{2ck'(0)}{2-c}\ (Flip\text{-}flop\ curve)^{17}, \qquad (15)$$

or as *Neimark-Hopf bifurcation*, when it surpasses the curve $a = \frac{c\left[\beta_p k'(0) - 1\right]}{\beta_p(1-c)}\ (Hopf\ curve).$ *curve*). (16)

In a situation which is evolving under the assumed fixed value $\beta_p > 0$ it is necessary to bear in mind that the shape of stability area in parametrical space *(c, a)*, is to a great extent under the strong impact of chartists' demand in steady state of (*k´(*0*)*).[18] Notably in case when ($\beta_p k'(0) \leq 1$) space is rather wide, because curves are more oval and are mirror reflected to one another (see Fig. 1). Here it is

[15] With Trace in Algebra the sum of items of major diagonal of the quadratic set is understood.

[16] With these problems, Gumowski and Mira have been concerned for more years (see [5] , 1980, p.159), the same [6], also J. C. Cathala [1], or open port Scholarpedia in Internet, for others.

[17] Correctly, instead of figure 2, it may be some abstract parameter, to say φ, φÎ(1,2), so formula (15) will be changed to $a = \frac{\varphi}{\beta_p} + \frac{\varphi c k'(0)}{\varphi - c},\ (15')$. (15´). This formula then can be used to construct "Flip-flop curve" in Excel.

[18] Let us remember that for a particular function of chartists' demand k´(0) chosen earlier, so after derivation it will be k´(0) = γ/(1 + g2).

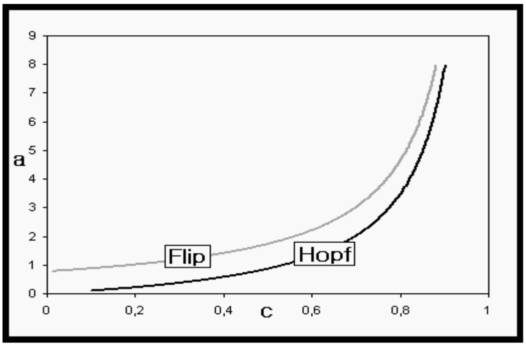
seen that when the potential of chartists is rather week, i.e. when ($k'(0)<1/\beta_p$), in the given level of reactive strength of chartists' (c) equilibrium is stable for sufficient low values of reactive parameter of fundamentalists, i.e. parameter a, but fundamentalists can cause instability if they react too strong to the deviation from fundamental value. In a subsequent graph (Fig. 2), we can see that if the strength of a characteristic demand is rather great, i.e. ($k'(0)>1/\beta_p$), the ability of fundamentalist demand to avoid instability of equilibrium is limited to rather low values of the fundamentalists' reaction parameter a, but fundamentalists can cause instability by reacting too strongly to the deviation from the fundamental value because curves are very steep and positively parallel. In the first case, the values were changed as follows: $\beta_p=2.6; \gamma=0.5; g=1; \beta_p k'(0)=0.65$. In the second one, values were changed like this: $\beta_p=2.6; \gamma=2.5; g=1; \beta_p k'(0)=3.25$.[19]

2.2. Passage: Invertibility of the map

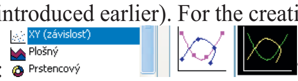
In special causes of parameters there can arise a situation that map T shall *noninvertible space*. That means that when it starts from certain initial values (let say as $(\psi_{0,1}, p_0)$) forward

Figure 2

A flip curve near to Hopf one



iteration of formula (12) uniquely defines the trajectory $(\psi_{t,t+1}, p_t) = T^t(\psi_{0,1}, p_0)$ $(t = 1, 2, \ldots)$, however backward iteration of that one, i.e. of (12) is not uniquely defined else. And really, the given point (ψ, p) of space may have several rank-1 preimages.[20] Let us assume, as we noted earlier that function $h(\cdot) = \gamma \arctan(\cdot)$, when $\gamma > 0$, is such, that $k(\psi) = \gamma \arctan(\psi - g) - \gamma \arctan(-g)$. In that situation it is possible to show by a simple geometrical argumentation that by defining

[19] For the reader who would like to independently construct these curves in Excel, we suggest this instruction: Into the left column we set sequence of numbers 0 £ ci £ 1.5, so that distance among them was as small as can be (because we want to reach smooth curve), say Dc=0.001. Into first cell of right column to set number 0 and into other cells we set formula (15) for creation curves of double period (Flip) and formula (16) for Hopf curve. For value of c in every row, we set number from the same row of left column and other parameters remains on (their values we are introduced earlier). For the creation of graph from these two columns, we are using the following routine: 

".

[20] If it is done, by mathematic mode, n-dimensional map F: Rn→Rn and it is done positive integer number r, then point y is preimage-r of a point x if F r(y) = x, i. e. if y is mapping into x in r iteration.

$$m = (a\beta_p - 1)\frac{(1-c)}{c}, \tag{17}$$

the given map has unique inversion for $m \le 0$ or for
$m \ge \gamma\beta_p$, but for $0 < m < \gamma\beta_p$ is map noninvertible.
In particular, by defining

$$\psi_1 = g - \sqrt{\frac{\gamma\beta_p}{m} - 1}, \quad q_1 = \beta_p k(\psi_1) - m\psi_1, \tag{18}$$

$$\psi_2 = g + \sqrt{\frac{\gamma\beta_p}{m} - 1}, \quad q_2 = \beta_p k(\psi_2) - m\psi_2, \tag{19}$$

Then point of (ψ, p) phase space for which function

$$q(\psi, p) = a\beta_p p - \frac{a\beta_p - 1}{c}\psi, \tag{20}$$

fulfils inequality $q(\psi, p)<q_1$, or other one $q(\psi, p)>q_2$ has unique rank-1 preimages, while the points for which the inequality is in interval $q_1<q(\psi, p)<q_2$ have three distinct rank-1 preimages. If we now fasten of notation introduced by the French mathematician C. Mira et al. [5, 6] for interval $0 <m<\gamma\beta_p$ that map is of type $Z_1–Z_3–Z_1$, which means that phase space is subdivided into different regions Z_j $(j = 1,... 3)$, and each point of which j has distinct rank-1 preimages. Such regions are bounded by the so-called *critical curves* of rank-1, defined as the locus of points having at least two merging rank-1 preimages as noted by the authors of ChiDiGar. These critical curves are defined as the locus of points having at least two merging rank-1 preimages (for details see works by Gumowski and Mira [5, 6]. For earlier mentioned map this set can be defined as

$$LC = \{(\psi, p) \in \Re^2 : q(\psi, p) = q_1 \cup q(\psi, p) = q_2\}, \tag{21}$$

where q_1 and q_2 are done in (18) and (19), and for that they are creating two straight line that is $LC_{-1} = L_{-1} \cup L'_{-1}$, where L and L′ have following equations

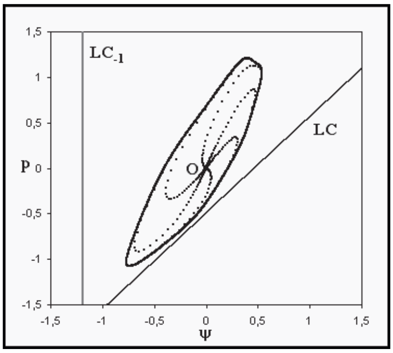
$$L : p = \frac{a\beta_p - 1}{a\beta_p c}\psi + \frac{q_1}{a\beta_p}; \qquad L' : p = \frac{a\beta_p - 1}{a\beta_p c}\psi + \frac{q_2}{a\beta_p}. \tag{22}$$

Each of these *critical points* $(\psi, p) \in LC$ has two *merging* rank-1 preimages, and the locus of such preimages, denoted by LC_{-1} (this defines the *critical curve* of rank-*0*), turns out to be made up of two straight lines, say $LC_{-1} = L_{-1} \cup L'_{-1}$, whose equations are

$$L_{-1} : \psi = g - \sqrt{\frac{\gamma\beta_p}{m} - 1}; \qquad L'_{-1} : \psi = g + \sqrt{\frac{\gamma\beta_p}{m} - 1}. \tag{23}$$

Figure 3

Two critical curves and the closed curve object



For the purpose of the first image, we created in Excel[21] the following snapshot having in mind situation (for their creation we used these values: b_p=2.6; *γ*=2.5; *a*=0.8; *c*=0.49; *m*=1.124082; and q_1»-0.15). Critical curve LC_{-1} in this place corresponding with locus of points of phase plane in which determinant of Jacobi set *DT(ψ, p)* vanished[22]. Images of set *LC* are called as c*ritical curves of higher ranks*, in particular for $LC_k = T^k(LC) = T^{k+1}(LC_{-1}),\ k=1,2,\ldots$ are called as *critical curves of rank -(k+1)* ($LC_0=LC$). While having in mind example these curves have also two branches so it is $LC_k = L_k \cup L'_k = T^{k+1}(L_{-1}) \cup T^{k+1}(L'_{-1})$. Let us return for a minute to the two earlier introduced snapshots of parametric spaces created in Excel, i.e. to *(c, a)* Fig. 1 and Fig. 2 for the purpose of comparison bifurcation curves with ranks of invertibility and/or noninvertibility of the map *T*. It is useful for understanding of those problems to sketch in the same s*pace* (*c, a*) the area in which there is fulfilled condition of noninvertibility of $0 < m < \gamma b_p$. If we bring into account equation (17) condition of $m > 0$ can be rewritten as $0 < c < 1, a > 1/b_p$, whereas condition $m < \gamma b_p$ is changed to $0 < c < 1, a < \gamma c/(1-c) + 1/b_p$. Noninvertibility space of map *T* was realised in Fig. 4 by dove-colour. The hatching is the area that is featured local stability region for origin *O*. It is needed to note however, that the higher is the value of *γ*, i.e. the strength of chartist demand at the steady state *k´(0),* the wider is the non-invertibility region. Moreover, we can see that in the region defined by $0 < c < 1,\ \ a > 1/b_p$, the map is invertible no matter what the value of *k´(0)*.

For reaching the correspondence with Fig. 1, and Fig. 2, we prepared also the second case, which can be seen in snapshot from Excel Fig. 5. The values were as in case of such as in Fig. 1, and in Fig. 2 also here: In the first case – b_p=2.6; *γ*=0.5; *g*=1; $b_p k'(0)$=0.65, and in the second one they were changed as follows - b_p=2.6; *γ*=2.5; *g*=1; $b_p k'(0)$=3.25.

[21] While the graph of trajectory and/or map of bifurcation of single parameter can be integrated without difficulties in Excel too, with other images, for example, with basin of attraction it may be more demanding. For that purpose, it is more confortable to use a routine from the environment of iDMC software for the creation of basin of attraction.

[22] In general, for a differentiable map is a critical curve LC−1 subset of loci of phase space points for which Jacobi set determinant of the given map is vacated.

So much for the demonstration of some mathematical part, which we borrowed from the

Figure 4

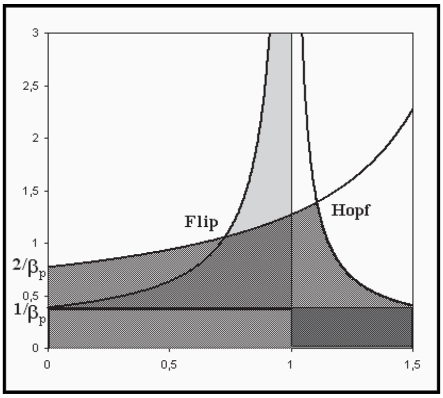
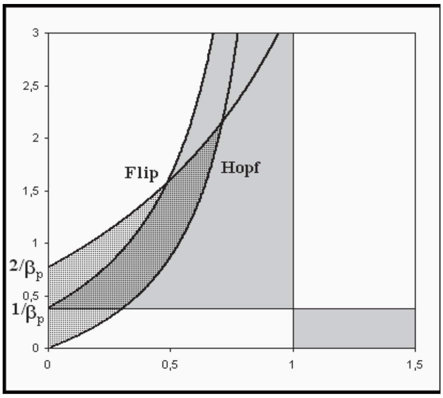


Figure 5



paper [2], and partially also (only a little) customized for our purposes. We consider, that all of these successfully show high demands of the problematic concerned. For higher figurativeness, we used some of possibilities which are offered by Excel program as well. If it were not for this possibility, our tasks would be more difficult.

Figure 6

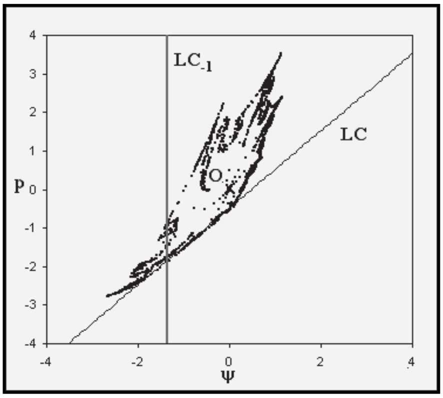
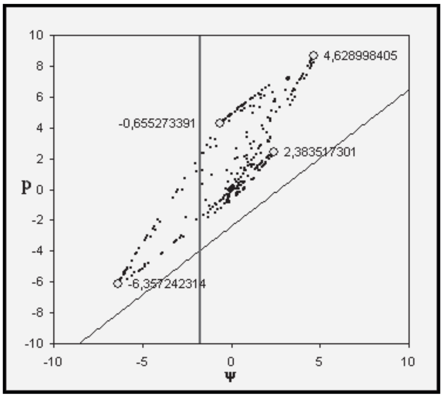
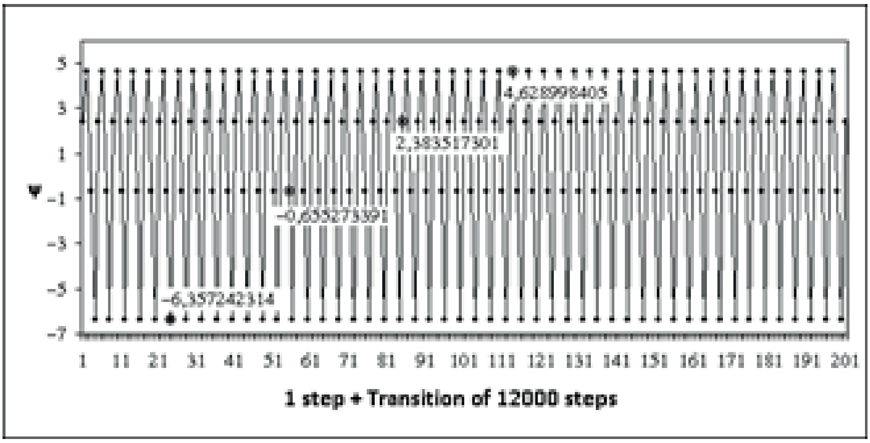


Figure 7



For the demonstration of importance of *critical curves* tool, we created in Excel again further snapshots of curves and attractors with the same parameters as before. In this place it is the visualisation of what kind of qualitative regimes appears when the parameters are changed. In Fig. 6 we can see a situation, which emerged after increasing the value of c parameter on $c = 0.52$ (in Fig. 3 that one was $c = 0.49$). This higher level of c shows a dynamical behaviour from the outer side of the stability area O, in the right from *Neimark-Hopf bifurcation curve*. If we go even to a higher level of c, namely to level $c = 0.6$ then, after a great number of iterations, the curve vanishes, and after some interval, chaos the solutions begin to jump to four constant valued points (in the cycle of four periods). Orbit in a snapshot of Fig. 7 has 4 periods, or cycles, which are regularly repeated as shown in Fig. 8, where it is a snapshot of iteration in time steps. The orbit depicted emerges only after a very great number of iteration namely cca after 12 000 iterations;[23] however, after that the interval it remains stable.

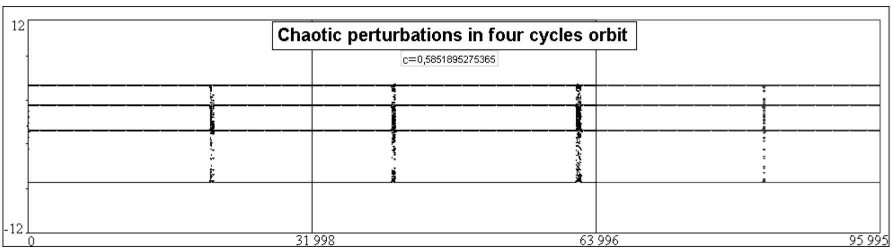
Figure 8



In lower values of *c* that is lower than *c = 0.6* at some time however certain number of determinist chaos iterations jump up. This phenomenon it is possible to visualize. Because Excel is allow in creation of graphs only 31 999 rows however, under greater number of iteration we must use further new columns. For creation of snapshot in Fig. 9 we are need in this case three independent columns each with 31 998 iterations. We can see very specific mode of this model behaviour. All this part of our essay used a lot of information, knowledge and tools cut-off from ChiDiGar end it is admirable example of visualization of mathematical difficulties of model concerned analyses

Figure 9

Time steps evolution



[23] So that the figures in horizontal axis are not too wide, we subtracted from transitions 12,000 steps, i. e. “1” in horizontal axis actually = 12,001 iterations.

3. Laboratory of Capital Market "Living" in iDMC Environment: Creating Phenotype

Although experimentation in laboratories built in Excel has certain usefulness too, but there are also some difficulties with their routines. For example, certain graphics needed for deeper understanding of behaviour of complex models are very work-intensive or some of them cannot be done. That is the reason why there is the need for using other ways and means. For example, on creation of *basins of attraction of main variables*, on building of *cycles* graphics, and/or for creation *basins of attraction of double parameters*, it is very useful and advantageous to work in laboratories created in iDMC environment and use several their routines. In the following part we are introduce two laboratories created by former descripted models, methods and tools.

3.1. Laboratory of market exclusively with (or without safe acquisitions) risk capital

Whereas in reality is a market with exclusively risk capital more complex case, in model form it may be dealing with them a little more simple than with safe acquisitions present in the market. This is the reason why we at first introduced a model written in iDMC vocabulary as:

Block 1

```
--@@
name = "Market of securities - Andrášik"
description = "Essay of this author ..."
type = "D"
parameters = {"a", "b", "c", "gama"}
variables = {"Y", "p"}
function f(a, b, c, gama, Y, p)
   Y1          =          (1-a)*Y-a*b*(c*p-
gama*math.atan(Y))
   p1 = p-b*(c*p-gama*math.atan(Y))
   return Y1, p1
end
function Jf(a, b, c, gama, Y, p)
   return
   1-a+a*b*gama/(1+Y^2), -a*b*c,
   p*gama/(1+Y^2), 1-b*c
```

In the laboratory referred to, we made several experiments, which can help deeper understanding of that model behaviour. In the snapshot of Fig. 10 we can see the basin of attraction after finishing the whole experiment, in which however the point of attractor is missing, because colours are not used there. Under a direct experimentation however, we can see all the process not only in colour but also in dynamics of evolution, so in this way it is possible to easier come into the essence of a complex behaviour of model running in iDMC. But also the black and white snapshot of experimentation results clearly show the complex behaviour of that model.

Of special advantage is the possibility of choosing only results of one period from calculation of very large steps of iteration for the visualisation of behaviour, which enables us to see what qualitative shape the solution is attracted to. In the case concerned, that contour is orbit with four cycles as we can see in a snapshot of Fig. 11. From the snap of evolution, it is easy to read that from instable fixed point *O* the process reaches that orbit after several steps of iteration, and it remains here forever. On the other side, the basin of attraction of Fig. 10 also intuitively suggests: the orbit is reachable from every loci of space (*Y, p*), but only upon

Figure 10

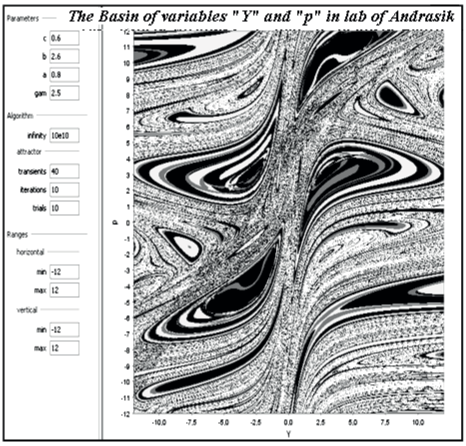
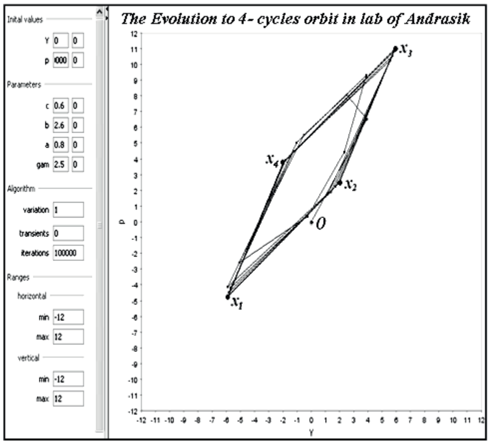


Figure 11



explicit trajectories from inner and from outer sides, and only if the number of iterations is very great the cycles are wholly stable as shown in Fig. 10. The snapshot of Fig. 11 shows possibilities of uncovering numbers of periods of orbits and/or the identification of chaos under different values ratio of parameters *c* and *a*. Numbers of orbits are denoted by figures next to the colours in em-quad bottom.[24] These ones can be verified in routine *Trajectory* afterwards.
The script of program in language LUA (under Java) in laboratory, which we created in environment iDMC inspired by the *ChiDiGar* model analysed earlier follows on page 14.

Figure 12

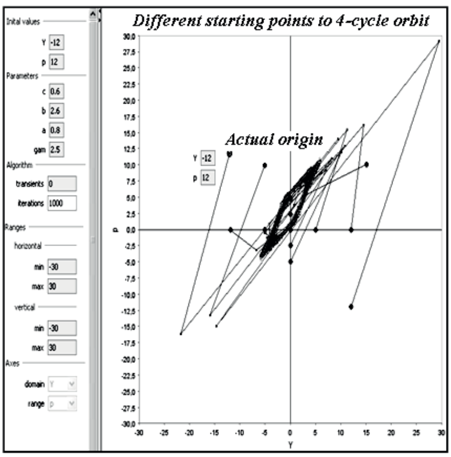
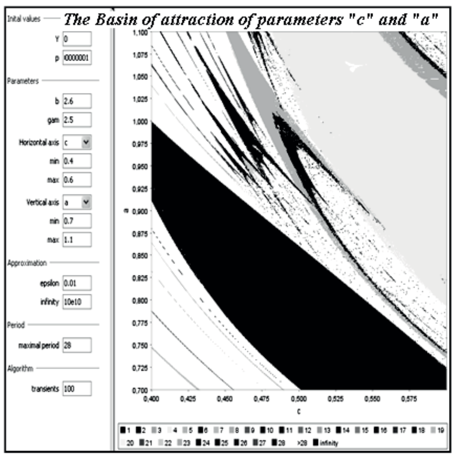


Figure 13



[24] Likewise in this case, the disability of traditional knowledge mining from black and white printings on paper is apparent. However, in experimenting in virtual laboratory, which we created in iDMC, it is possible to precisely control the number of cycles of given orbit and/or the case of orbit or the chaos.

3.2. Market laboratory with safe/risk capital – running model in iDMC be a phenotype

Similarly as in the preceding analysed case, also in this one we conducted several experiments in the given laboratory. We credit a great cognitive importance running experiments in routine *Basin of attraction* used with evolving value of two parameters. In the subsequent four snapshots of Fig. 14 and 15 there are results of experiments, whereby a reader has opportunity to compare these snapshots among one to others and convince oneself of the fact that under experimentation have possibility of seeing also into very deep moreover microscopic level of parameters basin.

Block 2

```
--@@
name = "Model of capital market inspired by Chiarella at all."
description = "See essay Andrášik"
type = "D"
parameters = {"c", "beta", "a", "gama", "g"}
variables = {"Y", "p"}
function f(c, beta, a, gama, g, Y, p)
Y1 = (1-c)*Y-c*beta*(a*p-(gama*math.atan(Y-g)-gama*math.atan(-
g)))
p1 = p-beta*(a*p-(gama*math.atan(Y-g)-gama*math.atan(-g)))
return Y1, p1
end
function Jf(c, beta, a, gama, g, Y, p)
return
1-c+c*beta*gama*((1/(1+(Y-g)^2))-1/(1+(-g)^2)), -c*beta*a,
```

Figure 14

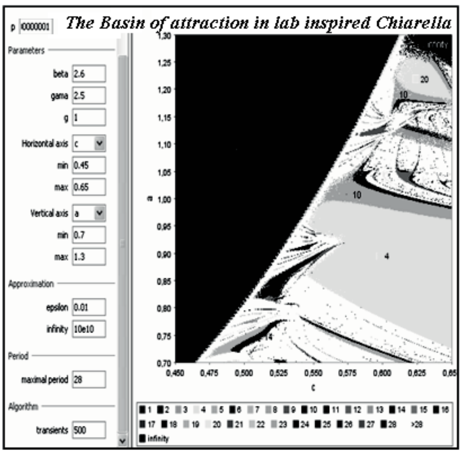


Figure 15

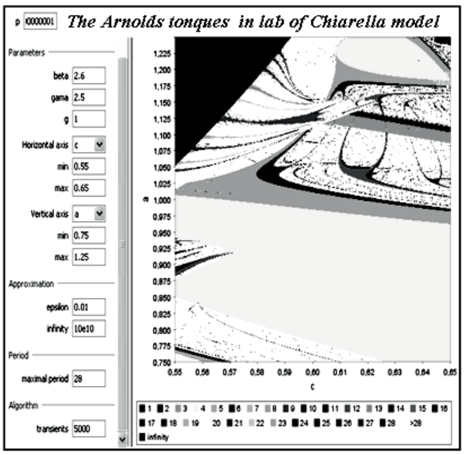
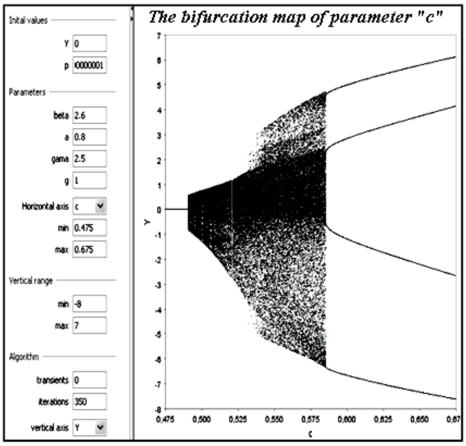


Figure 16



18

Figure 17

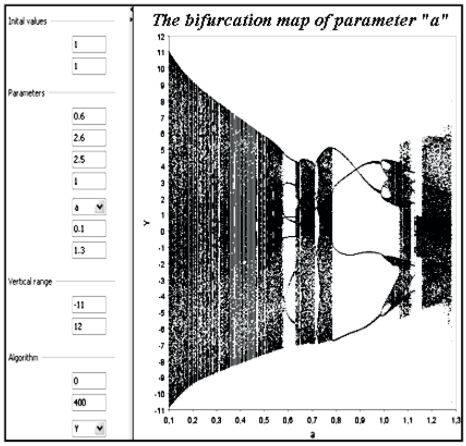
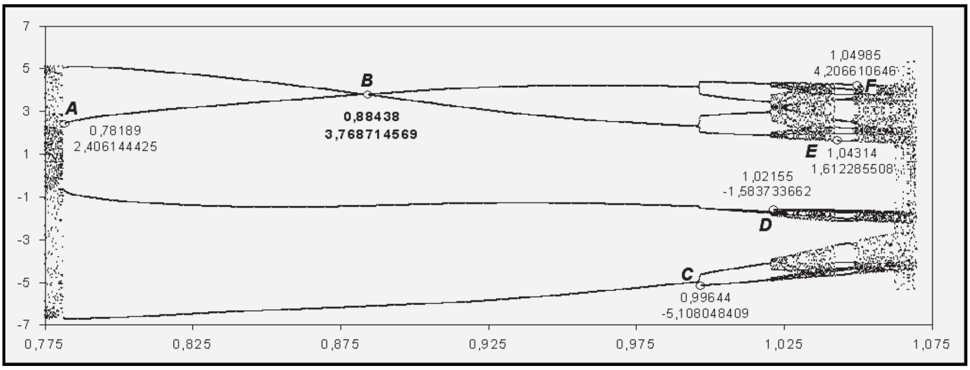


Figure 18



In the snapshot of fig. 18 we denoted by capital letters some of the important points that are six explicit values of parameter (a), at which important incidents of bifurcation happen. In the point sign as A the chaos has *vanished* and evolution jumps to four-period orbit.

Figure 19

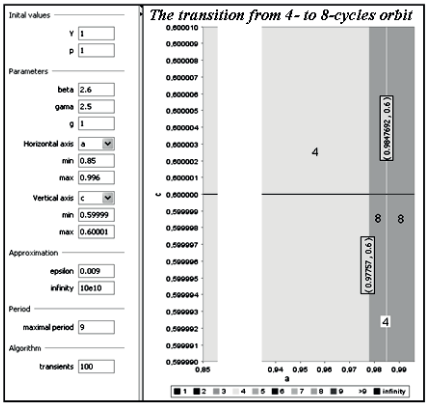
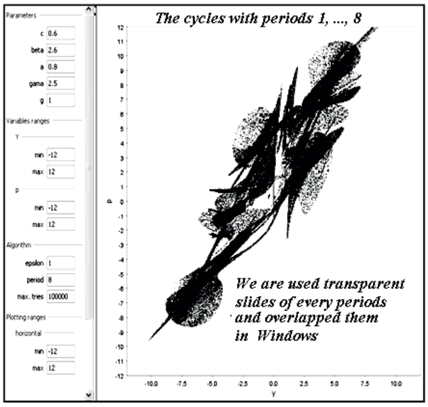


Figure 20



Extraordinary interesting and unique event happens in point *B*, where in singular value of parameter (*a*)[25] there are borne 3-period orbits. The snapshot on Fig. 19 shows ambiguous transition from four-cycle to eight-cycle orbit. This transition was conducted "at two strokes", to put it metaphorically. In the snapshot of Fig. 20, we present the possibilities of imaging in the "Cycles" routine. It enables to represent points in the surroundings of single periods, if they exist in the solving of arrangement (12). In the snapshots of Fig. 21 and 22, we show the possibilities of *absorption areas*.

Figure 21

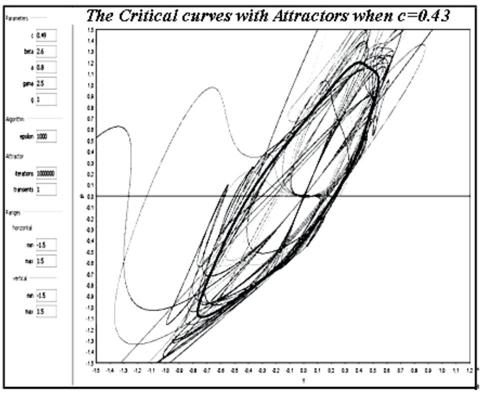
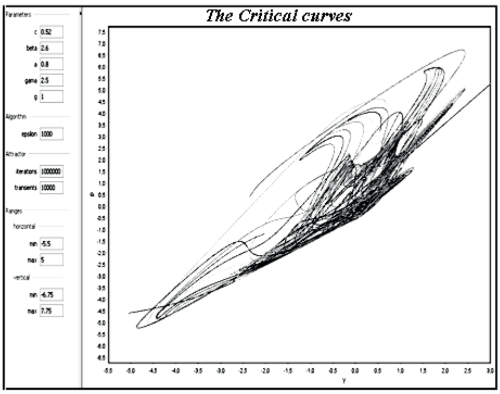


Figure 22



[25] In order to be correct, we have to note that the values of (*a*) parameter are not quite exact in the snapshot as they were detected "only manually". That is of special importance in B case as in that case it is a completely unique value.

Figure 23

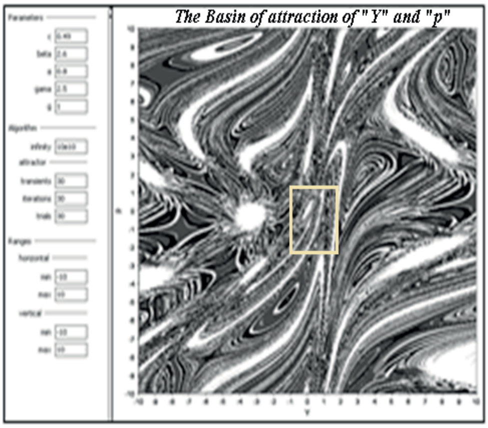
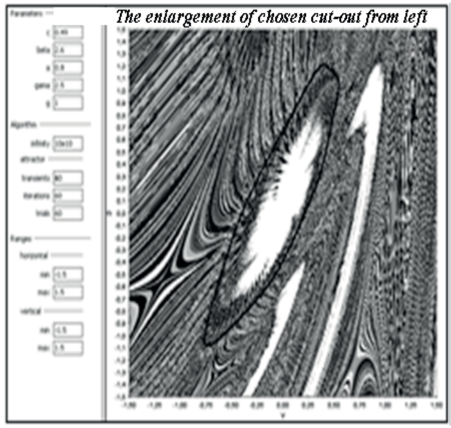


Figure 24



The snapshots on Fig. 23-24 prove the possibilities of seeing into "entrails" of basins of attraction of the main variables. In the snapshot of Fig. 23, there is basic snapshot, which is then "microscoped". Firstly, we made a "magnifier" Fig. 24, to visualise the attractor, on this basis it is then comparable with what is depicted in Fig. 23. In Fig. 25 there is even a deeper incursion into the interior of the capital market model.

Figure 25

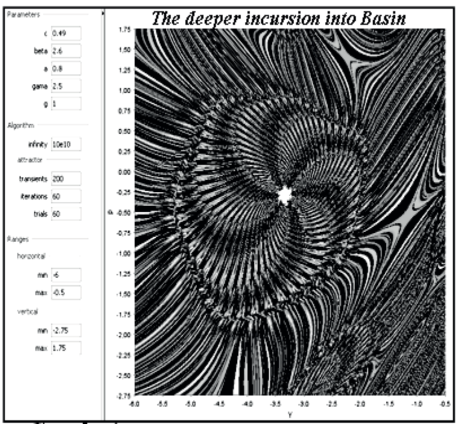
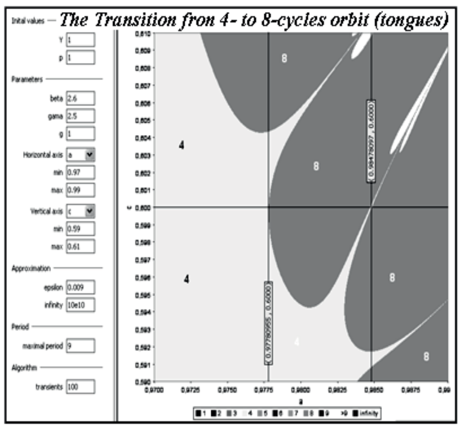


Figure 26



Conclusions

A careful reader has certainly realised that the problem of this essay is in numerous snapshots, which occupy a great space. This resulted naturally in a smaller space for verbal explanation. However, on the other hand, we must again emphasize that the focus is just on these snapshots from experiments. It may be ideal evidently, if interested scholar and or student can directly experiment in our virtual laboratories. Unfortunately, the black printing on paper does not allow us more than only these "frozen" images. These may be metaphorically regarded as some type of “successions” in iterative step, in which adaptation process is “frozen”. In other words, this was the main goal of the paper, too: to show that there are possibilities of experimentation in virtual laboratories, which help us achieve a deeper knowledge of *complex economic process*. The chosen voluminous economic model fulfils only a secondary plan: it has to persuade that this holds for also in the case of relatively complex tasks, as is definitely the capital market.

With this aim in mind, we decided to create a model for investigating the dynamics of price of securities in the capital market in discrete time. Further goal was to show that the experimentation in virtual laboratories could contribute to better understanding of complex dynamics, which emerge in the market in securities, because of different modes of behaviour of traders and also persons who lack good mathematical skills. On this basis, a space is openings for a wider community of economists in gaining knowledge also in similar complexity problems, and in this way, a kind of some “democratization” of cognitive processes in economics takes place. On the basis of theoretical and methodology starting points, we mainly worked with the studies of earlier cited authors [2, 3]. The model is formulated on the mutual interactions of two populations of security traders in the capital market. They differ from one another in terms of the mode of creating a plan for their decision making on selling and/or buying securities.

In the economic literature, one type of traders is called *fundamentalists* and the other one as *chartists*. Fundamentalists base their decisions on the estimation of base value and current price of securities. In our text, this demand is named as *p*.

The chartists differ from them by founding their trading decisions on the analysis of former trends of prices and their demand is formed by shape of letter “S” (graph of function of this shape is generated by goniometric function *arcus tangent*), that is, by the function of expectation of earning disparately to alternative security certificates. In consensus with authors of [3], we introduce positions of the market supervisor (or a market maker), whose mission is to push the excess demand to null and reach in this way the clearing of the market at the end of every trading interim (or of every trading day) with regards to the compensation of long and short positions in the money market. The market maker does his work in the way described by the author cited as follows: (i) at the beginning of day *t* the market maker announces the (log) price P_t for that day; (ii) the market participants then form excess demand D_t according to (9) i. e. $D_t = a(W - P_t) + h(\psi_{t,t+1} - g)$ (see also [2], p. 177); (iii) the market maker, observing the excess demand, takes a long or short position M_t (by adjusting his or her inventory of assets) in order to clear the market, i.e. such that $D_t + M_t = 0$; (iv) the market maker then announces, at the beginning of the next trading period, the new (log) price P_{t+1} calculated as the previous (log) price plus some fraction of the excess demand of the previous period, according to, $P_{t+1} = P_t + \beta_p D_t \left(\beta_p > 0\right)$ at the end of trading day. We presented two types of mutually interacting behaviour of traders and their consequences for the dynamics of the market by showing the impact on global and local dynamics by means of two key parameters. The traders use these key parameters to manipulate the market, i.e. potential demand of the chartists (Υ)

and fundamentalists (*a*) and by the third parameter, which expresses the adjusting velocity of chartists' expectations (*c*). In particular, we focused on the impact of chartists' demand potential (*ϒ*) on the local stability of equilibrium, i.e. under a reasonable low value of (*ϒ*) the equilibrium is stable also if the change of parameter (*a*) value is very wide of *(a)* fundamentalists and any value of reactive velocity of the chartists (*c*), i.e. ($0 < c \leq 1$). And vice versa, when the velocity (*ϒ*) is adequately high and the chartists' demand is relatively high, the ability of fundamentalists to stabilise an overall dynamics is enough only for a narrow interval of values (*a*). Chiarella and co-workers concentrated their attention just on this circumstance of a dynamic behaviour, and we also did a lot of experiments, of which we could unfortunately show only few because they needed an excessive place. It is remarkable too that in a situation when the equilibrium is locally stable, there may emerge other dynamic phenomena under adequately large values of parameters (*c*) and (*a*), such as chaotic transitions before the convergence into stable equilibrium or into coexistence attractors.[26] We showed too that with the assistance of laboratory created in iDMC and partially also in Excel, it is possible to conduct detailed analyses of global dynamics phenomena, which are emerging when we are manipulating with the value of parameter (*c*). Fixed point *O* is becoming instable through *Neimark-Hopf bifurcation*. By experimenting with parameters (*c*) and (*a*), we detected chaotic behaviour around point *O* and under very large values we discovered the disappearance of chaos and the transition to orbit with a small number of periods. Due to a limited scope of the paper, we could not insert more snapshots from results of our experimentations, which we wanted to use, at least for the purpose of visual impression also other results conducted in accordance to proper routines of iDMC (*cycles, critical curves*, and *Ljapunov exponents*). Despite those decisions, we are convinced that a reader could gain an adequate idea of the possibilities with experimentation in virtual laboratories in this case with relatively simple and easy to manage software resources. Mainly this is the prerequisite for a rising number of people who are able to increase their knowledge about the behaviour of complex and evolving partial economic systems by means of these methods and tools.

[26] Also in this place we encourage the reader to learn how to "read" graphical results of virtual experiments and consistently "read" such snapshot as for example map of bifurcation on the snapshot of Fig. 15 and moreover in Fig. 16. Essential and very important qualitative changes for our understanding of complex phenomena are visible there very clearly in values of parameter (*a*). We point out that such bifurcation map can be constructed independently in Excel too. It is possible to deduct an explicit value of parameter (a) on such created graph, under which emerge "explosion" or "vanishing" cases. (We created Fig. 18 in Excel).

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